Journal of Statistical Physics, Vol. 111, Nos. 1/2, April 2003 (© 2003)

New Developments in the Eight Vertex Model

Klaus Fabricius[1] **and Barry M. McCoy**[2]

Received July 19, 2002; accepted October 16, 2002

We demonstrate that the Q matrix introduced in Baxter's 1972 solution of the eight vertex model has some eigenvectors which are not eigenvectors of the spin reflection operator and conjecture a new functional equation for $Q(v)$ which both contains the Bethe equation that gives the eigenvalues of the transfer matrix and computes the degeneracies of these eigenvalues.

KEY WORDS: Bethe's ansatz; loop algebra; quantum spin chains.

1. INTRODUCTION

Thirty years ago Baxter[1] computed the eigenvalues of the transfer matrix of the eight vertex model in a paper of unsurpassed brilliance and creativity. One of the key steps of this method is the invention of an auxiliary matrix $Q(v)$ which satisfies a functional equation with the transfer matrix $T(v)$.

One year later Baxter computed the eigenvectors of the transfer matrix[2–4] and in the course of that computation he again obtains the functional equation between $T(v)$ and $Q(v)$ previously derived in ref. 1. However, the definitions of $Q(v)$ used in ref. 1 and in refs. 2–4 are not the same and Baxter comments that (page 15 of ref. 2) "The above methods provide a different (though obviously related) definition of $Q(v)$ to that of ref. 1 which may help us to understand $Q(v)$ a little better."

We have been interested in extending our studies[5–8] of the degeneracies of the spectrum of the transfer matrix of the six vertex model at roots of unity to the eight vertex model and have seen in many numerical

[1] Physics Department, University of Wuppertal, 42097 Wuppertal, Germany; e-mail: Fabricius @theorie.physik.uni-wuppertal.de

[2] Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794-3840; e-mail: mccoy@insti.physics.sunysb.edu

0022-4715/03/0400-0323/0

examples that the exponentially degenerate multiplets of the six vertex model also exist in the eight vertex model. In the course of the search for an explanation of these degeneracies which would extend the sl_2 loop algebra symmetry of the six vertex model to some analogous algebraic structure for the eight vertex model we have examined these two definitions of $Q(v)$ in detail. We have discovered that while the two definitions are obviously related that the two $Q(v)'$ s so defined are in fact different. First of all we find that there are cases where $Q_{72}(v)$ does not exist. Furthermore in the case where $Q_{72}(v)$ exists we have found that all the eigenvectors of $Q_{73}(v)$ of refs. 2–4 and ref. 9 are eigenvectors of the spin reflection operator whereas some of the eigenvectors of the $Q_{72}(v)$ defined in ref. 1 are not. This lack of invariance under spin reversal of $Q_{72}(v)$ does not affect the computation of the eigenvalues of $T(v)$ of ref. 1 but it does affect their degeneracy. Furthermore we have conjectured a functional equation for $Q_{72}(v)$ which incorporates the "Bethe equation" whose roots specify the eigenvalues of $T(v)$ and which also computes the degeneracy of these eigenvalues by demonstrating that $Q_{72}(v)$ has zeroes specified by the function introduced recently by Deguchi.[(10–11)]

In Section 2 we review the formalism of ref. 1. In Section 3 we present our conjectured functional equation. We close in Section 4 with a discussion of the functional equation and its significance.

2. FORMALISM OF THE EIGHT VERTEX MODEL

We use the notation of Baxter's 1972 paper.[(1)] The transfer matrix for the eight vertex model with N columns and periodic boundary conditions is

$$T_8(u)|_{\mu,\nu} = \mathrm{Tr}\, W_8(\mu_1, \nu_1)\, W_8(\mu_2, \nu_2) \cdots W_8(\mu_N, \nu_N) \tag{2.1}$$

where in the conventions of (6.2) of ref. 1

$$\begin{aligned} W_8(1, 1)|_{1,1} &= W_8(-1,-1)|_{-1,-1} = \rho\Theta(2\eta)\,\Theta(v-\eta)\,H(v+\eta) \\ W_8(-1,-1)|_{1,1} &= W_8(1, 1)|_{-1,-1} = \rho\Theta(2\eta)\,H(v-\eta)\,\Theta(v+\eta) \\ W_8(-1, 1)|_{1,-1} &= W_8(1,-1)|_{-1,1} = \rho H(2\eta)\,\Theta(v-\eta)\,\Theta(v+\eta) \\ W_8(1,-1)|_{1,-1} &= W_8(-1, 1)|_{-1,1} = \rho H(2\eta)\,H(v-\eta)\,H(v+\eta). \end{aligned} \tag{2.2}$$

The definition and useful properties of $H(v)$ and $\Theta(v)$ are recalled in the appendix.

The computations of ref. 1 are restricted to values of η which satisfy the "root of unity condition" (C15) of ref. 1. We here further restrict our attention to the case which will connect with our previous computations[5–8] in the six vertex model by setting in (C15) $m_2 = 0$ and thus obtaining

$$2L\eta = 2m_1 K. \tag{2.3}$$

In the 1972 paper[1] Baxter defines a matrix $Q_{72}(v)$ and states on p. 200 of ref. 1 that "... there are two elementary ways in which the matrix $T(v)$, and hence the matrix $Q_{72}(v)$, can be broken up into diagonal blocks or subspaces" which are characterized by the quantum numbers ν' and ν'' of (6.4) and (6.5) of ref. 1. The transfer matrix $T(v)$ certainly has this block diagonalization property and if the transfer matrix $T(v)$ were non degenerate the same property would follow for $Q_{72}(v)$ because $Q_{72}(v)$ commutes with the transfer matrix. But in the root of unity case (2.3) the transfer matrix has degenerate eigenvalues and in these degenerate subspaces the eigenvectors of $Q_{72}(v)$ can fail to be eigenvectors of the spin reflection operator.

In ref. 1 the matrix $Q_{72}(v)$ is explicitly defined by (C37)

$$Q_{72}(v) = Q_R(v)\, Q_R^{-1}(v_0) \tag{2.4}$$

where v_0 is an arbitrary normalization point at which $Q_R(v)$ is nonsingular. The matrix $Q_R(v)$ in (2.4) is defined as

$$[Q_R(v)]_{\alpha\,|\,\beta} = \mathrm{Tr}\, S(\alpha_1, \beta_1)\, S(\alpha_2, \beta_2) \cdots S(\alpha_N, \beta_N) \tag{2.5}$$

where α_j and $\beta_j = \pm 1$ and $S(\alpha, \beta)$ is an $L \times L$ matrix given as (C16),

$$S(\alpha, \beta) = \begin{pmatrix} z_0 & z_{-1} & 0 & 0 & \cdot & 0 \\ z_1 & 0 & z_{-2} & 0 & \cdot & 0 \\ 0 & z_2 & 0 & z_{-3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & z_{1-L} \\ 0 & 0 & 0 & \cdot & z_{L-1} & z_L \end{pmatrix} \tag{2.6}$$

with (C17)

$$z_m = q(\alpha, \beta, m\,|\,v) \tag{2.7}$$

and (C19)

$$\begin{aligned} q(+, \beta, m\,|\,v) &= H(v+K+2m\eta)\, \tau_{\beta, m}, \\ q(-, \beta, m\,|\,v) &= \Theta(v+K+2m\eta)\, \tau_{\beta, m} \end{aligned} \tag{2.8}$$

and we recall from (2.3) that

$$\eta = m_1 K/L. \tag{2.9}$$

The $\tau_{\beta,m}$ are generically arbitrary but we note that if they are all set equal to unity then $Q_R(v)$ is so singular that its rank becomes 1. On the other hand as long as the $\tau_{\beta,m}$ are chosen so that there is a v_0 such that $Q_R(v_0)$ is not singular then $Q_{72}(v)$ is independent of $\tau_{\beta,m}$.

In ref. 1 it is stated that $Q_R(v)$ is nonsingular for generic values of v for any L for $N=1, 2$. We have extended these studies of the rank of $Q_R(v)$ up to $N=9$ and $L=17$ and have found that the nonsingularity of $Q_R(v)$

Table I. Rank of the Matrix $Q_R(v)$ for Generic Values of v as a Function of L, m_1, and N. The Ranks of the Matrices Which Are Singular Are Marked in Bold Face

		rank					
L	m_1	$N=2$	$N=4$	$N=6$	$N=7$	$N=8$	$N=9$
2	1	4	16	64	128	256	512
3	1	4	16	64	128	256	512
	2	**3**	**7**	**18**	**29**	**47**	**76**
4	1	4	16	64	128	256	512
	3	4	16	64	128	256	512
5	1	4	16	64	128	256	512
	2	4	**13**	**38**	**57**	**117**	**193**
	3	4	16	64	128	256	512
	4	4	**13**	**38**	**57**	**117**	**193**
6	1	4	16	64	128	256	512
	5	4	16	64	128	256	512
7	1	4	16	64	128	256	512
	2	4	16	**57**	**64**	**187**	**247**
	3	4	16	64	128	256	512
	4	4	16	**57**	**64**	**187**	**247**
	5	4	16	64	128	256	512
	6	4	16	**57**	**64**	**187**	**247**
8	1	4	16	64	128	256	512
	3	4	16	64	128	256	512
	5	4	16	64	128	256	512
	7	4	16	64	128	256	512
9	1	4	16	64	128	256	512
	2	4	16	64	**64**	**248**	**256**
	4	4	16	64	**64**	**248**	**256**
	5	4	16	64	128	256	512
	7	4	16	64	128	256	512
	8	4	16	64	**64**	**248**	**256**

breaks down if L is odd and m_1 is even for sufficiently large N. The results for $2 \leqslant L \leqslant 9$ are presented in Table I.

There are several features of our study for $N \leqslant 9$ and $L \leqslant 17$ to be explicitly noted.

(1) For L even and for L odd and m_1 odd the matrix $Q_R(v)$ is generically nonsingular

(2) For L odd, m_1 even, and N even $Q_R(v)$ is singular if $N \geqslant L-1$. For $L=3$ and $N=2$ this contradicts the statement on nonsingularity on p. 218 of ref. 1.

(3) For L odd, m_1 even and N odd $Q_R(v)$ is singular for all L. For $L \geqslant N$ the rank is 2^{N-1} which is one half the dimension of the matrix.

(4) For even N and all L $Q_R(v)$ is singular at $v=0, K, iK$ and $K+iK'$; for odd N and all L $Q_R(v)$ is singular at $v=K$ and $K+iK'$ but not at 0 and iK'; for even N and $L>2$ $Q_R(v)$ is also singular at $v= \pm\eta$.

The method of ref. 1 assumes that $Q_R(v)$ is nonsingular and hence cannot literally hold in the cases where L is odd and m_1 is even. However when N is even we may use the symmetry of the transfer matrix eigenvalues $t(v;\eta)$

$$t(v+K; K-\eta) = (-1)^{\nu'}\, t(v;\eta) \tag{2.10}$$

(where $\nu' = 0, 1$ is the quantum number (6.4) of ref. 1) to study the singular case with m_1 even by transforming to the case $m_1 \to L-m_1$ where $Q_R(v)$ is nonsingular. In the rest of this paper we restrict our attention to the cases where $Q_{72}(v)$ exists.

From the definition the eigenvectors of $Q_{72}(v)$ may be explicitly computed for small values of N and L. We have done this for $L=2, 3$, $m_1=1$ and $N=8$ and found that $Q_{72}(v)$ is non degenerate and that in the subspaces where the eigenvalues of $T(v)$ are degenerate there are eigenvectors of $Q_{72}(v)$ which are not eigenvectors of the spin reflection operator.

The failure of the eigenvectors of $Q_{72}(v)$ to all be eigenvectors of the spin reflection operator means that the quantum number ν'' of ref. 1 can not in general be used for all eigenvectors of $Q_{72}(v)$. Therefore instead of the transformation properties (6.9) of ref. 1 we have

$$Q_{72}(v+2K) = (-1)^{\nu'}\, Q_{72}(v) \tag{2.11}$$

$$Q_{72}(v+2iK') = q^{-N} \exp(-iN\pi v/K)\, Q_{72}(v) \tag{2.12}$$

where we note that (2.12) follows from the identity

$$S(\alpha, \beta \mid v+2iK') = q^{-1}e^{-i\pi v/K}MS(\alpha, \beta \mid v)\, M^{-1} \tag{2.13}$$

with

$$M_{j,j'} = e^{-\pi i\eta j(j-1)/K}\delta_{j,j'}. \tag{2.14}$$

From (2.11) and (2.12) one derives the most general form for the eigenvalues of $Q_{72}(v)$ to be

$$Q_{72}(v) = \mathscr{K}(q; v_k)\exp(-i\nu\pi v/2K) \prod_{j=1}^{N} H(v-v_j) \tag{2.15}$$

where

$$e^{i\pi(\nu'+\nu+N)} = 1 \qquad \text{so} \quad \nu'+\nu+N = \text{even integer} \tag{2.16}$$

$$e^{\pi i(-i\nu K'/K+N+\sum_{j=1}^{N} v_j/K)} = 1 \qquad \text{so} \quad N+\left(-\nu iK'+\sum_{j=1}^{N} v_j\right)\Big/K = \text{even integer} \tag{2.17}$$

and $\mathscr{K}(q; v_k)$ is a normalization constant independent of v. We choose by convention that the v_j lie in the fundamental region

$$0 \leqslant \text{Re } v_j \leqslant 2K, \qquad 0 \leqslant \text{Im } v_j \leqslant 2K.' \tag{2.18}$$

The values of the even integers in the sum rules depend on the choice of these conventions.

From the imaginary part of (2.17) we find an explicit formula for ν

$$\nu = \sum_{j=1}^{N} \text{Im } v_j/K' \tag{2.19}$$

and using this in (2.16) we find

$$\nu = \sum_{j=1}^{N} \text{Im } v_j/K' = \text{even integer} - \nu' - N. \tag{2.20}$$

From the real part of (2.17) we obtain the sum rule

$$N + \sum_{j=1}^{N} \text{Re } v_j/K = \text{even integer} \tag{2.21}$$

We note that the difference between the form (2.15) and the form (6.10) of ref. 1 is that the imaginary period of the fundamental region of (6.10) is exactly half that of the fundamental region of (2.15).

For the cases $\eta = K/2, K/3$ $(L = 2, 3)$ and $N = 8$ the values of v_j in (2.15) have been determined numerically and we find that not only are the eigenvalues of $Q_{72}(v)$ all of the form (2.15) but in fact can be written in the form

$$Q_{72}(v) = \mathscr{K}(q; v_k) \exp(-i\nu\pi v/2K) \prod_{j=1}^{n_B} H(v - v_j^B)\, H(v - v_j^B - iK')$$
$$\times \prod_{j=1}^{n_L} H(v - iw_j)\, H(v - iw_j - 2\eta) \cdots H(v - iw_j - 2(L-1)\,\eta) \tag{2.22}$$

where

$$2n_B + Ln_L = N, \tag{2.23}$$

n_L is even, the w_l are real, from (2.20) ν is given by

$$\nu = n_B + \left(L \sum_{j=1}^{n_L} w_j + 2 \sum_{j=1}^{n_B} \operatorname{Im} v_j^B \right) \Big/ K' = \text{even integer} - \nu' - N \tag{2.24}$$

and from (2.21) the v_j^B satisfy the sum rule

$$N + n_L(L-1) + 2 \sum_{j=1}^{n_B} \operatorname{Re} v_j^B / K = \text{even integer}. \tag{2.25}$$

We conjecture that the form (2.22) is correct for all even N but we have explicitly seen that for odd N it fails.

It is clear from (2.22) that there are two types of roots while the form (6.10) of ref. 1 incorporates only one of these two types. We remark that the zeros at

$$v = iw_j + 2l\eta \quad l = 0,\ldots, L-1 \tag{2.26}$$

are not the same as the zeroes of the form

$$v_j^B = v_j + 2l\eta \tag{2.27}$$

which have been called[12] complete L-strings. The zeroes of the form (2.26) have not previously been seen.

Baxter shows in ref. 1 that the transfer matrix satisfies a functional equation (4.5) with (6.3)

$$T(v)\,Q_{72}(v)=[\rho h(v-\eta)]^{N}\,Q_{72}(v+2\eta)+[\rho h(v+\eta)]^{N}\,Q_{72}(v-2\eta) \tag{2.28}$$

where

$$[T(v),Q_{72}(v')]=[T(v),T(v')]=[Q_{72}(v),Q_{72}(v')]=0 \tag{2.29}$$

and

$$h(v)=\Theta(0)\,\Theta(v)\,H(v) \tag{2.30}$$

with

$$h(v+2K)=-h(v) \tag{2.31}$$

$$h(v+iK')=q^{-1/2}e^{-\pi iv/K}h(v). \tag{2.32}$$

When our form (2.22) is put into (2.28) we see that all the dependence on the w_l cancels out and we are left with a "Bethe's equation" for the v_j^B

$$\left(\frac{h(v_l^B-\eta)}{h(v_l^B+\eta)}\right)^{N}=e^{2\pi ivm_1/L}\prod_{\substack{j=1\\ l\neq j}}^{n_B}\frac{h(v_l^B-v_j^B-2\eta)}{h(v_l^B-v_j^B+2\eta)} \tag{2.33}$$

where v is given by (2.24), the v_j^B obey the sum rules (2.24) and (2.25) and in the phase factor we have used the root of unity condition (2.3). When $n_L=0$ this reduces to equation (10.6.10) of Baxter's book[9] and this equation is commonly called "Bethe's" equation. For this reason we call the v_j^B Bethe roots. We note that for $n_L\neq 0$ that (2.33) has the same form as Baxter's (10.6.10) but that both the v given by (2.24) which appears in the phase factor in (2.33) and the sum rule in (2.24) depend on both v_j^B and w_j whereas the phase (10.6.7a) and the sum rule (10.6.7b) of ref. 9 depends only on v_j^B.

3. FUNCTIONAL EQUATION

To complete the specification of the eigenvalues of $Q_{72}(v)$ we need to compute the w_l. We wish to do this by producing a functional equation satisfied by $Q_{72}(v)$. We take our inspiration from

(1) The polynomial $Y(v)$ (1.42) of our paper[8]

$$Y(v) = \sum_{l=0}^{L-1} \frac{\sinh^N \frac{1}{2}(v-(2l+1)\, i\gamma_0)}{Q^B(v-2il\gamma_0)\, Q^B(v-2i(l+1)\,\gamma_0)} \tag{3.1}$$

where

$$Q^B(v) = \prod_{k=1}^{n} \sinh \tfrac{1}{2}(v - v_k^B) \tag{3.2}$$

where the v_k^B are the "ordinary" Bethe roots which do not include any of the complete strings. It is useful to rewrite this in the form

$$\begin{aligned} &\frac{Y(v)\, Q^B(v)\, Q^B(v-2i\gamma_0)}{\sinh^N \frac{1}{2}(v-i\gamma_0)} \\ &\quad = \sum_{l=0}^{L-1} \frac{\sinh^N \frac{1}{2}(v-(2l+1)\, i\gamma_0)}{\sinh^N \frac{1}{2}(v-i\gamma_0)} \frac{Q(v)\, Q(v-2i\gamma_0)}{Q(v-2il\gamma_0)\, Q(v-2i(l+1)\,\gamma_0)} \end{aligned} \tag{3.3}$$

where the $Q(v)$ on the right is the full $Q(v)$ including all the complete strings. The complete strings may be included here because they cancel between the numerator and denominator.

(2) The function of Deguchi in (5.8) of ref. 10 and (31) of ref. 11 which may be written with $r = 0$ as

$$\begin{aligned} G(v) = &\sum_{l=0}^{L-1} e^{-4\eta c l} \frac{h^N(v-(2l+1)\,\eta)}{h^N(v-\eta)} \\ &\times \prod_{k=1}^{n_B} \frac{h(v-v_k^B)\, h(v-v_k^B-2\eta)}{h(v-v_k^B-2l\eta)\, h(v-v_k^B-2(l+1)\,\eta)} \end{aligned} \tag{3.4}$$

where the v_k^B are the ordinary roots and do not include complete strings. This may easily be rewritten in terms of $Q_{72}(v)$ as given in (2.22) as

$$G(v) = \sum_{l=0}^{L-1} \frac{h^N(v-(2l+1)\,\eta)}{h^N(v-\eta)} \frac{Q_{72}(v)\, Q_{72}(v-2\eta)}{Q_{72}(v-2l\eta)\, Q_{72}(v-2(l+1)\,\eta)} \tag{3.5}$$

(3) The functional equation for the transfer matrix T_q of the 3 state chiral Potts model (2.21) of ref. 13

$$T_q T_{Rq} T_{R^2q} = K\left[\left(\frac{ab}{cd}\eta^2-1\right)^N\left(\frac{ab}{dc}\eta^2\omega^2-1\right)^N T_q \right.$$
$$+\left(\frac{ab}{cd}\eta^2\omega^2-1\right)^N\left(\frac{ab}{cd}\eta^2\omega-1\right)^N T_{R^2q}$$
$$\left.+\left(\frac{ab}{cd}\eta^2-1\right)^N\left(\frac{ab}{cd}\eta^2\omega-1\right)^N T_{R^4q}\right] \tag{3.6}$$

where R is the automorphism (1.20) of ref. 13 (which is not to be confused with the spin reflection operator). This is better written by dividing by $T_q T_{R^2q} T_{R^4q}$ where the general form of (3.6) for general integer L is

$$\frac{T_{Rq}}{T_{R^{2(L-1)}q}} = K\sum_{l=0}^{L-1}\frac{f_l^N}{T_{R^{2l}q}T_{R^{2(l+1)}q}} \tag{3.7}$$

with

$$f_l = \left(\frac{ab}{cd}\eta^2\omega^{-l}-1\right)\left(\frac{ab}{cd}\eta^2\omega^{1-l}-1\right) \tag{3.8}$$

The form of the right hand side of (3.7) is in the same form as the right hand side of (3.1). If we divide by the term with $l=L-1$ we obtain a form with a right hand side comparable to (3.3) and (3.5)

$$\frac{T_{Rq}T_q}{f_{L-1}^N} = K\sum_{l=0}^{L-1}\frac{f_l^N}{f_{L-1}^N}\,\frac{T_{R^{2(L-1)}q}T_q}{T_{R^{2l}q}T_{R^{2(l+1)}q}}. \tag{3.9}$$

In order to make a conjecture we note that the right hand side of (3.5) will agree with (3.9) if we replace the automorphism R^2 by the shift $v\to v-2K/L$. The only other reasonable replacement is to let in the left hand side of (3.9) the automorphism R be replaced by the automorphism $v\to v-iK'$. We also note that any conjecture must be invariant under the transformation $v\to v+2iK'$. Thus we are led to the following

Conjecture

For N even and either L even or L and m_1 odd

$$e^{-N\pi i v/2K}Q_{72}(v-iK')$$
$$= A\sum_{l=0}^{L-1} h^N(v-(2l+1)\,\eta)\,\frac{Q_{72}(v)}{Q_{72}(v-2l\eta)\,Q_{72}(v-2(l+1)\,\eta)} \tag{3.10}$$

where A is a normalizing constant matrix independent of v that commutes with Q_{72}. What this matrix is depends on the normalization value of v_0 in the definition (2.4) of Q_{72}.

There are several points to be noted about this conjecture.

(1) The exponential factor in the left hand side is needed to maintain invariance under the transformation $v \to v+2iK'$ as can be seen by use of (2.32) and (2.12). It is also needed to insure that both sides are invariant under $v \to v+2K$ by use of (2.31) and (2.11). With this factor both sides of the conjectured functional equation are quasi periodic functions with the same fundamental region.

(2) If we multiply out the denominators this conjecture may be rewritten as

$$\begin{aligned} &e^{-N\pi iv/2K} Q_{72}(v-iK') \prod_{l=1}^{L-1} Q_{72}(v-2l\eta) \\ &\quad = A \Bigg\{ \sum_{l=0}^{L-2} h^N(v-(2l+1)\,\eta)\, Q_{72}(v) \\ &\qquad \cdots Q_{72}(v-2(l-1)\,\eta)\, Q_{72}(v-2(l+2)\,\eta) \\ &\qquad \cdots Q_{72}(v-2(L-1)\,\eta) \\ &\qquad +(-1)^{\nu'} h^N(v-(2L-1)\,\eta) \prod_{l=1}^{L-2} Q_{72}(v-2l\eta) \Bigg\} \end{aligned} \tag{3.11}$$

In this form the conjecture has been proven for $L=2$ and all even N and has been numerically verified for $L=3$, $m_1=1$ and $N=8$.

(3) If we use (2.22) in (3.10) we obtain

$$\begin{aligned} &(-1)^{n_B+n_L}\, q^{(\nu/2-n_B)} \exp\left(-\frac{i\pi}{K} \left\{ (\nu-n_B+N/2)\, v + \sum_{j=1}^{n_B} v_j^B \right\} \right) \\ &\quad \times \prod_{j=1}^{n_L} \prod_{l=0}^{L-1} H(v-iw_j-2l\eta)\, H(v-i(w_j+K')-2l\eta) \\ &\quad = A\mathscr{K}(q;v_k^B)^{-2} \sum_{l=0}^{L-1} \frac{e^{-iv\pi(2l+1)/L} h^N(v-(2l+1)\,\eta)}{\prod_{j=1}^{n_B} H(v-2l\eta-v_j^B)\, H(v-2l\eta-v_j^B-iK')} \\ &\quad \times \frac{1}{\prod_{j=1}^{n_B} H(v-2(l+1)\,\eta-v_j^B)\, H(v-2(l+1)\,\eta-v_j^B-iK')} \end{aligned} \tag{3.12}$$

Here we note that the left hand side depends only on w_l and the right hand side only on v_l^B.

(4) The apparent poles in (3.12) when $v = v_k^B + 2l\eta$, $v_k^B + 2l\eta + iK'$ cancel because the v_k^B are specified by the Bethe equation (2.33). This is exactly what we saw in the expression (3.1) for the polynomial $Y(v)$ in ref. 8. Equivalently we may say that the Bethe's equation for v_k^B is already included in (3.10).

(5) The left hand side of (3.12) is symmetric under exchange of w_l and $w_l + K'$ and all theta functions H appear in pairs $H(u)$, $H(u - iK')$ which can be combined to $h(u)$. Therefore for any given set of Bethe roots v_k^B the n_L equations in (3.12) have 2^{n_L} independent solutions for the w_l which thus determines the dimensionality of the multiplet of degenerate eigenstates of the transfer matrix.

(6) In the XXZ limit the right hand side of (3.12) reduces to the polynomial $Y(v)$ (1.42) of ref. 8 once the possibility of Bethe roots at infinity is taken into account. The zeroes of $Y(v)$ have been identified with the evaluation parameters of the loop sl_2 symmetry algebra of the XXZ model in ref. 8.

4. DISCUSSION

The conjectured functional equation (3.10) provides an elegant computation of the 2^{n_L} degeneracy of the eigenvalues of the transfer matrix of the eight vertex model previously found in our numerical computations. In the six vertex model limit these powers of 2 are explained by showing that only spin one half representations occur in the decomposition into irreducible representations of the loop sl_2 symmetry algebra. The functional equation (3.10) obtains this result without any reference to a symmetry algebra. Such a derivation of the eigenvalue multiplicities is superior to the derivation from the loop algebra because the loop algebra symmetry of the six vertex model has only been analytically demonstrated[(5)] for the case $S^z \equiv 0 \pmod{L}$.

In the above we have written $Q_{72}(v)$ in terms of the $Q_R(v)$ by use of the first equation in (C37) of ref. 1. In order to use the corresponding formulas with $Q_L(v)$ the right hand sides of the two equations in (C26) need to be interchanged. This modification of (C26) is consistent with the subsequent equations in appendix C.

At present the functional equation (3.10) is a conjecture which requires proof. Because of the similarity of (3.10) to the functional equation of the chiral Potts model it is natural to investigate whether the techniques used in the chiral Potts model can be extended to the eight vertex model. The proof of the chiral Potts functional equation is given in the work of Bazhanov and

Stroganov[14] and Baxter, Bazhanov, and Perk.[15] However, it is instantly apparent from these papers that there exists the possibility that there may be even more matrices beyond $Q_{72}(v)$ and $Q_{73}(v)$ which satisfy Baxter's functional equation (2.28). In particular for the six vertex model is shown on p. 805 on ref. 14 that there is a five parameter family of $Q(v)$ matrices (denoted by $\mathscr{T}$) which satisfy Baxter's functional equation and at the very least it seems plausible that both $Q_{72}(v)$ and $Q_{73}(v)$ should be embedded in such a larger family. This reflects the fact that all methods of solution of the 8 vertex model include steps which, while they are sufficient to obtain the transfer matrix eigenvalues, are quite probably not necessary. As an example we note that the condition

$$[Q(v), Q(v')] = 0 \tag{4.1}$$

required by Baxter[1–4] for both $Q_{72}(v)$ and $Q_{73}(v)$ can be replaced by the weaker condition

$$[Q(v), Q(v \pm 2\eta)] = 0 \tag{4.2}$$

which is necessary to reduce the functional equation for the matrices $T(v)$ and $Q(v)$ (2.28) to an equation for eigenvalues. The work of ref. 14 shows that matrix $\mathscr{T}$ satisfies (4.2) but that in general (4.1) does not hold. It is surprising that after 30 years these questions have not been resolved.

APPENDIX A: THETA FUNCTIONS

The definition of Jacobi Theta functions of nome q is

$$H(v) = 2 \sum_{n=1}^{\infty} (-1)^{n-1} q^{(n-\frac{1}{2})^2} \sin[(2n-1)\pi v/(2K)] \tag{A.1}$$

$$\begin{aligned} \Theta(v) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(nv\pi/K) \\ &= -iq^{1/4} e^{\pi i v/(2K)} H(v+iK') \end{aligned} \tag{A.2}$$

where K and K' are the standard elliptic integrals of the first kind and

$$q = e^{-\pi K'/K}. \tag{A.3}$$

These theta functions satisfy the quasi periodicity relations (15.2.3) of ref. 9

$$H(v+2K) = -H(v) \tag{A.4}$$

$$H(v+2iK') = -q^{-1} e^{-\pi i v/K} H(v) \tag{A.5}$$

and

$$\Theta(v+2K) = \Theta(v) \tag{A.6}$$

$$\Theta(v+2iK') = -q^{-1}e^{-\pi iv/K}\Theta(v). \tag{A.7}$$

From (A.2) we see that $\Theta(v)$ and $H(v)$ are not independent but satisfy (15.2.4) of ref. 9

$$\begin{aligned} \Theta(v+iK') &= iq^{-1/4}e^{-\frac{\pi iv}{2K}}H(v) \\ H(v+iK') &= iq^{-1/4}e^{-\frac{\pi iv}{2K}}\Theta(v). \end{aligned} \tag{A.8}$$

ACKNOWLEDGMENTS

We are pleased to thank Prof. R. J. Baxter, Prof. V. V. Bazhanov, and Prof. V. Jones for fruitful discussions. One of us (BMM) is pleased to thank the Miller Institute and the mathematics department of the University of California at Berkeley for their hospitality. This work is partially supported by NSF Grant DMR 0078058.

REFERENCES

1. R. J. Baxter, Partition function of the eight vertex model, *Ann. Phys.* **70**:193 (1972).
2. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain I: Some fundamental eigenvectors, *Ann. Phys.* **76**:1 (1973).
3. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain II: Equivalence to a generalized Ice-type lattice model, *Ann. Phys.* **76**:25 (1973).
4. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain III: Eigenvectors of the transfer matrix and the Hamiltonian, *Ann. Phys.* **76**:48 (1973).
5. T. Deguchi, K. Fabricius, and B. M. McCoy, The sl_2 loop algebra symmetry of the six-vertex model at roots of unity, *J. Stat. Phys.* **102**:701 (2001).
6. K. Fabricius and B. M. McCoy, Bethe's equation is incomplete for the XXZ model at roots of unity, *J. Stat. Phys.* **103**:647 (2001).
7. K. Fabricius and B. M. McCoy, Completing Bethe's equations at roots of unity, *J. Stat. Phys.* **104**:573 (2001).
8. K. Fabricius and B. M. McCoy, Evaluation parameters and Bethe roots for the six vertex model at roots of unity, in *Progress in Mathematical Physics*, Vol. 19, M. Kashiwara and T. Miwa, eds. (Birkhauser, Boston, 2002), p. 119.
9. R. J. Baxter, *Exactly Solved Models* (Academic Press, London, 1982).
10. T. Deguchi, Construction of some missing eigenvectors of the XYZ spin chain at discrete coupling constants and the exponentially large spectral degeneracy of the transfer matrix, *J. Phys. A* **35**:879 (2002).
11. T. Deguchi, The 8V CSOS model and the sl_2 loop algebra symmetry of the six-vertex model at roots of unity, *Int. J. Mod. Phys. B* **16**:1899 (2002).

12. R. J. Baxter, On the completeness of the Bethe Ansatz for the six and eight-vertex models, *J. Stat. Phys.* **108**:1 (2002).
13. G. Albertini, B. M. McCoy, and J. H. H. Perk, Eigenvalue spectrum of the superintegrable chiral Potts model, *Adv. Stud. Pure Math.* **19**:1 (1989).
14. V. V. Bazhanov and Yu. G. Stroganov, Chiral Potts model as a descendant of the six-vertex model, *J. Stat. Phys.* **59**:799 (1990).
15. R. J. Baxter, V. V. Bazhanov, and J. H. H. Perk, Functional relations for transfer matrices of the chiral Potts model, *Int. J. Mod. Phys. B* **4**:803 (1990).